CALL OPTION PRICES BASED ON BESSEL PROCESSES

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Abstract. As a complement to some recent work by Pal and Protter [9], we show that the call option prices associated with the Bessel strict local martingales are integrable over time, and we discuss the probability densities obtained thus.

1. Introduction: some general remarks

1.1. Let $(M_t, t \geq 0)$ denote a continuous local martingale, taking values in $\mathbb{R}_+$. To any $K > 0$, we associate the process $(K - M_t)^+,\ t \geq 0$. It is not difficult to show, after localizing M, that this process $(K - M_t)^+$ is a (bounded) submartingale, and, as a consequence, the function:

$$m_K^{(+)}(t) = E\big[\,(K - M_t)^+\,\big],\ t \geq 0$$

is increasing, and bounded (by K). The study of such functions, considered (essentially) as distribution functions, has been the subject of the Bachelier Course [1, 2], given by the second author. In particular, if $M_t \underset{t\to\infty}{\longrightarrow} 0$, there is the formula

$$E\left[F_t\,(K - M_t)^+\right] = KE\left[F_t 1_{(\mathcal{G}_K \leq t)}\right] \tag{1}$$

which is valid for every $F_t \geq 0$, $(\mathcal{F}_t)$ measurable, and $\mathcal{G}_K = \sup\{t : M_t = K\}$. See also Madan-Roynette-Yor [6, 7, 8].

1.2. The present paper is devoted to the study of the functions:

$$m_K^{(-)}(t) = E\big[\,(M_t - K)^+\,\big] = E\big[\,(K - M_t)^-\,\big],\ t \geq 0,$$

which play an important role in option pricing, as $(m_K^{(-)}(t))$ is the European call price with strike K, and maturity t, associated with the local martingale (M_t). If (M_t) is a "true" martingale, then $\{(M_t - K)^+\}$ is a submartingale, hence $(m_K^{(-)}(t), t \geq 0)$ is increasing. On the other hand, if $(M_t, t \geq 0)$ is a strict local martingale, that is: a local martingale, which is not a martingale, then the function $(m_K^{(-)})$ is not in general increasing, or even monotone.

1.3. The most well-known example of a strict local martingale is $M_t = 1/R_t$, where $(R_t, t \geq 0)$ denotes the $BES(3)$ process, starting from 1, or, by scaling, equivalently from any $r > 0$. Then, the study of $(m_K^{(-)}(t))$ in this particular case has been undertaken in a remarkable paper by S. Pal and P. Protter [9]; the results of which have strongly motivated the present paper.

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In the present paper, we take up again the study of this function $(m_K^{(-)}(t))$ in this particular case; we show that:

$$\int_0^\infty dt\ m_K^{(-)}(t) < \infty.$$

Hence, up to a multiplicative constant $(m_K^{(-)}(t),\ t \geq 0)$ is a probability density on $\mathbb{R}_+$; we identify the Laplace transform of this probability, and describe it as the law of a certain random variable defined uniquely in terms of $BES(3)$ process. This is done thanks to the Doob h-transform understanding of $BES(3)$ (from Brownian motion, killed when hitting 0), combined with general identity (1). We refer the reader to Section 2 for precise statements. In Section 3, we develop the same kind of study but this time with $M_t = 1/R_t^{(\delta-2)},\ t \geq 0$, where $(R_t,\ t \geq 0)$ denotes the $BES(\delta)$ process, starting from 1. In Section 4, we present the graphs of the corresponding functions $(m_K^{(-)}(t),\ t \geq 0)$.

1.4. To summarize, the main point of this work is to use the interpretation of the generalized Black-Scholes quantities in terms of last passage times (formula (1)) in the framework of Bessel processes in order to derive fine properties of the call option process, as a function of maturity, written for the strict local Bessel martingales.

2. Some results about $r_K^{(3)}(t) \equiv E_1^{(3)}\left[\left(\frac{1}{X_t} - K\right)^+\right],\ t \geq 0$

2.1. In this section, we change notation slightly: $(X_t,\ t \geq 0)$ denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}_+)$, W_x is Wiener measure such that $W_x(X_0 = x) = 1$, and $P_x^{(3)}$ is the law of the $BES(3)$ process starting from x. In fact, we shall only consider $x = 1$ (except mentioned otherwise).

2.2. Here are our 3 main results concerning the functions $r_K^{(3)}(t)$.

Proposition 1. *The following holds:*

(i) $(t \to \infty)$*:* $r_0^{(3)}(t) \sim \sqrt{\frac{2}{\pi t}}$; $\ r_K^{(3)}(t) \sim \frac{1}{3\sqrt{2\pi} t^{3/2} K^2}$

(ii) $(t \to 0)$*:* $r_K^{(3)}(t) \to (1-K)^+$*;* $r_1^{(3)}(t) \sim \sqrt{\frac{t}{2\pi}}$.

An important consequence of Proposition 1, (i), is that, for $K > 0$, the function $(r_K^{(3)}(t), t \geq 0)$ is integrable over $\mathbb{R}_+$, hence it is, up to a multiplicative constant, a density of probability on $\mathbb{R}_+$. We now describe this probability.

Proposition 2.

(i) *The function* $(3K^2 r_K^{(3)}(t), t \geq 0)$ *is a probability density on* $\mathbb{R}_+$.

(ii) *It is the density of*

$$\Lambda_K^{(3)} \overset{\text{(law)}}{=} (g_1 - \widetilde{T}_k) + \widetilde{T}_k U \tag{2}$$

where, on the RHS of (2), $k = 1/K$*, the variables* g_1*,* $\widetilde{T}_k$*, and* U *are independent,* $g_1 = \sup\{t : X_t = 1\}$*,* $\widetilde{T}_k$ *is the size-biased sampling of* $T_k = \inf\{t : X_t = k\}$[1]*, with* g_1 *and* T_k *defined with respect to* $P_0^{(3)}$*, and finally* U *is uniform on* $[0,1]$.

As a further description of the law of $\Lambda_K^{(3)}$, we present its Laplace transform.

[1]That is: $\widetilde{T}_k$ satisfies $E[f(\widetilde{T}_k)] = 3K^2 E[f(T_k)T_k]$ for every $f : \mathbb{R}_+ \to \mathbb{R}_+$, Borel.

Proposition 3. *The Laplace transform of $\Lambda_K^{(3)}$ is: (we use again $k = 1/K$)*

$$E\left[\exp(-\lambda\Lambda_K^{(3)})\right] = \frac{3}{\lambda k^2}\left(e^{-\sqrt{2\lambda}}\right)\left(\frac{\sinh(\sqrt{2\lambda}k)}{\sqrt{2\lambda}k} - 1\right)$$

2.3. **Proofs of Propositions 1, 2, 3.**

2.3.1. The main ingredients of these proofs are the following:

(i) the Doob h-process relationship between Brownian motion and $BES(3)$, which may be written as:

$$P_1^{(3)}|_{\mathcal{F}_t} = (X_{t\wedge T_0}) \cdot W_1|_{\mathcal{F}_t}$$

(ii) the particular instance of formula (1) with $M_t = X_{t\wedge T_0}$, under W_1:

$$W_1\left(F_t\left(k - X_{t\wedge T_0}\right)^+\right) = kW_1\left(F_t 1_{(\gamma_k \le t)}\right)$$

with $\gamma_k = \sup\{t < T_0 : X_t = k\}$.

(iii) the time reversal result: $(X_{T_0 - t},\ t \le T_0)$ under W_1 is distributed as $(X_t,\ t \le g_1)$ under $P_0^{(3)}$.

2.3.2. Thanks to the preceding points, we may now obtain interesting description of $r_K^{(3)}(t)$ in terms of first and last passage times. In fact, we obtain:

$$r_K^{(3)}(t) = W_1\left(\gamma_k < t < T_0\right) \tag{3}$$

or, equivalently, from the time reversal result in (iii):

$$r_K^{(3)}(t) = P_0^{(3)}(g_1 > t) - P_k^{(3)}(g_1 > t). \tag{4}$$

Proof of (3):
Combining (i) and (ii) above, we obtain:

$$\begin{aligned} r_K^{(3)}(t) &= W_1\left((1 - KX_t)^+ 1_{(t<T_0)}\right) \\ &= KW_1\left((k - X_{t\wedge T_0})^+ 1_{(t<T_0)}\right), \quad \text{with } k = \frac{1}{K} \\ &= KkW_1\left(\gamma_k \le t < T_0\right) \\ &\equiv W_1\left(\gamma_k < t < T_0\right), \quad \text{which is (3).} \end{aligned}$$

2.3.3. Proof of Proposition 2:

a) We deduce from (3) that:

$$\int_0^\infty dt\ r_K^{(3)}(t) = W_1\left(T_0 - \gamma_k\right) = E_0^{(3)}(T_k),$$

by time reversal. Using the fact that $(R_t^2 - 3t, t \ge 0)$ is a $P_0^{(3)}$-martingale, we get:

$$3E_0^{(3)}(T_k) = k^2.$$

Hence, the constant c we were seeking is: $c = \frac{3}{k^2}$, and $\frac{3}{k^2} r_K^{(3)}(t) \equiv 3K^2 r_K^{(3)}(t)$ is a probability density on $\mathbb{R}_+$.

b) In order to identify a random variable $\Lambda_K^{(3)}$ with distribution $3K^2 r_K^{(3)}(t)$, we go back to (3) and we get, for any $f : \mathbb{R}_+ \to \mathbb{R}_+$, Borel:

$$\int_0^\infty dt\ f(t)\ 3K^2 r_K^{(3)}(t) = 3K^2 W_1 \left(\int_{\gamma_k}^{T_0} dt\ f(t) \right)$$
$$= 3K^2 W_1 \left[f\left(\gamma_k + (T_0 - \gamma_k)\, U\right) (T_0 - \gamma_k) \right]$$
$$= 3K^2 E_0^{(3)} \left[f\left((g_1 - T_k) + T_k U\right) T_k \right],$$

where U is uniform on $[0,1]$, independent of T_k and g_1. Hence using the notation of $\widetilde{T}_k$ for the size-biased sampling of T_k, we get that:

$$E\left(f\left(\Lambda_K^{(3)}\right)\right) = E_0^{(3)} \left[f\left(\left(g_1 - \widetilde{T}_k\right) + \widetilde{T}_k U \right)\right].$$

□

2.3.4. Proof of Proposition 3:
From formula (3) again, this Laplace transform is:

$$3K^2 W_1 \left(\int_{\gamma_k}^{T_0} dt\ e^{-\lambda t} \right) = \frac{3K^2}{\lambda} W_1 \left(e^{-\lambda \gamma_k} - e^{-\lambda T_0} \right).$$

The result now follows from:

$$W_1\left(e^{-\lambda T_0}\right) = e^{-\sqrt{2\lambda}} = W_1\left(e^{-\lambda\gamma_k}\right) \frac{\sqrt{2\lambda}k}{\sinh(\sqrt{2\lambda}k)}.$$

□

2.3.5. Proof of Proposition 1:

(i) As $t \to \infty$, we have:

$$r_0^{(3)}(t) = E_1^{(3)}\left(\frac{1}{R_t}\right) = \frac{1}{\sqrt{t}}\ E_{(1/\sqrt{t})}^{(3)}\left(\frac{1}{R_1}\right)$$
$$\underset{t\to\infty}{\sim} \frac{1}{\sqrt{t}}\ E_0^{(3)}\left(\frac{1}{R_1}\right) = \sqrt{\frac{2}{\pi t}}.$$

The equivalent, for $K > 0$, of $r_K^{(3)}(t)$, as $t \to \infty$, is simply the particular case: $\delta = 3$ of the result in (i) of Proposition 4.

(ii) The first statement follows from the convergence in L^1 of $\frac{1}{X_t}$ to 1 as $t \to 0$; For the second statement, we use:

$$r_1^{(3)}(t) = W_1\left((1 - X_t)^+\, 1_{(t<T_0)} \right)$$
$$= W_1\left((1 - X_t)^+ \right) - W_1\left((1 - X_t)^+\, 1_{(T_0<t)} \right).$$

The first term is: $W_0(X_t^+) = \sqrt{\frac{t}{2\pi}}$, whereas the second one may be majorized by

$$W_1\left(T_0 < t\right) = W_0\left(T_1 < t\right) = W_0\left(|X_1| > \frac{1}{\sqrt{t}}\right)$$
$$= \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{t}}^\infty du\ e^{-(u^2/2)} \le e^{-1/2t}$$

(Note that, in fact, it is not necessary to use Proposition 1 to obtain Propositions 2 and 3; however, it is the result of Proposition 1, (i), which led us to study $\Lambda_K^{(3)}$).

2.3.6. A more direct proof of the estimate: $r_K^{(3)}(t) = \mathrm{O}\left(\frac{1}{t^{3/2}}\right)$, as $t \to \infty$.
Since this estimate plays quite some role in our paper, it seems of some interest to look for a simple proof of it. We note that:

$$r_K^{(3)}(t) \equiv E_1^{(3)}\left(\left(\frac{1}{R_t} - K\right)^+\right) \leq E_0^{(3)}\left(\left(\frac{1}{R_t} - K\right)^+\right)$$

since, from the additivity property of squares of Bessel processes (Shiga-Watanabe [14]), a Bessel process with dimension δ, starting from $a > 0$, dominates stochastically a Bessel process with dimension δ, starting from 0. Now we have:

$$\begin{aligned} & E_0^{(3)}\left[\left(\frac{1}{R_t} - K\right)^+\right] \\ = \; & E_0^{(3)}\left[\left(\frac{1}{\sqrt{t}R_1} - K\right)^+\right] \qquad \text{(by scaling)} \\ = \; & c\int_0^{1/\sqrt{t}K} dr\ r^2 e^{-r^2/2}\left(\frac{1}{\sqrt{t}r} - K\right), \quad \text{for some universal constant c} \\ \leq \; & \frac{c}{\sqrt{t}}\int_0^{1/\sqrt{t}K} dr\ re^{-r^2/2} \\ = \; & \frac{c}{\sqrt{t}}\left(1 - \exp\left(-\frac{1}{2tK^2}\right)\right) \underset{t\to\infty}{\sim} \frac{c}{2t^{3/2}K^2}. \end{aligned}$$

3. Extending the previous results to: $r_K^{(\delta)}(t) \equiv E_1^{(\delta)}\left[\left(\frac{1}{R_t^{\delta-2}} - K\right)^+\right]$, $\delta > 2$

3.1. In this section, $(X_t,\ t \geq 0)$ still denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}_+)$, and we consider $P_x^{(\delta)}$ the law of the $BES(\delta)$ process, starting from x (which, again, will be taken mainly equal to 1).

3.2. As a parallel to Section 2, we offer 3 results concerning the function $r_K^{(\delta)}$. We note $\nu = \frac{\delta}{2} - 1$

Proposition 4. *The following holds:*

(i) $(t \to \infty)$*:* $r_0^{(\delta)}(t) \sim \frac{1}{t^\nu 2^\nu \Gamma(1+\nu)}$; $r_K^{(\delta)}(t) \sim \frac{C_K^{(\delta)}}{t^{\nu+1}}$*, with* $C_K^{(\delta)} = \frac{1}{2^{\nu+1}(\nu+1)\Gamma(\nu)K^{(1/\nu)}}$

(ii) $(t \to 0)$*:* $r_K^{(\delta)}(t) \to (1-K)^+$*;* $r_1^{(\delta)}(t) \sim \sqrt{t}(2\nu)\frac{1}{\sqrt{2\pi}}$

Proposition 5.

(i) *The function* $\left(\delta K^{2/(\delta-2)} r_K^{(\delta)}(t), t \geq 0\right)$ *is a probability density on* $\mathbb{R}_+$*.*

(ii) *It is the density of*

$$\Lambda_K^{(\delta)} \overset{\text{(law)}}{=} \left(g_1 - \widetilde{T}_k\right) + \widetilde{T}_k U \tag{5}$$

where, on the RHS of (5), $k = 1/K^{1/(\delta-2)}$*, the variables* g_1*,* $\widetilde{T}_k$*, and* U *are independent,* $g_1 = \sup\{t : X_t = 1\}$*,* $\widetilde{T}_k$ *is the size-biased sampling of*

$T_k = \inf\{t : X_t = k\}$, with g_1 and T_k defined with respect to $P_0^{(\delta)}$, and finally U is uniform on $[0,1]$.

Finally, we present the Laplace transform of $\Lambda_K^{(\delta)}$.

Proposition 6. *The Laplace transform of $\Lambda_K^{(\delta)}$ is:*

$$E\left[\exp(-\lambda\Lambda_K^{(\delta)})\right] = \frac{2\delta K^{2/(\delta-2)}}{\lambda}\left(\mathcal{K}_\nu(\sqrt{2\lambda})\right)\left\{\frac{\nu I_\nu(k\sqrt{2\lambda})}{k^\nu} - \frac{1}{\Gamma(\nu)}\left(\frac{\sqrt{2\lambda}}{2}\right)^\nu\right\}$$

where I_ν and $\mathcal{K}_\nu$ denote the usual modified Bessel functions, with parameter ν (we use $\mathcal{K}_\nu$ instead of K_ν so that no confusion with the strike K may occur).

3.3. **Proofs of Propositions 4, 5, 6.** We follow the rationale of the proofs of Propositions 1, 2, 3, after extending adequately the points (i), (ii) and (iii) in (2.3.1) from the $BES(3)$ process to $BES(\delta)$ process, for $\delta > 2$.

3.3.1. Here are these extensions:

(i)$_\delta$ the Doob h-process relationship between $BES(\delta)$ and $BES(4-\delta)$, killed upon hitting 0, is

$$P_1^{(\delta)}|_{\mathcal{F}_t} = (X_{t\wedge T_0})^{2\nu} \cdot P_1^{(4-\delta)}|_{\mathcal{F}_t},$$

where $\delta = 2(1+\nu)$.

(ii)$_\delta$ the particular instance of formula (1) with $M_t = (X_{t\wedge T_0})^{2\nu}$ under $P_1^{(4-\delta)}$:

$$E_1^{(4-\delta)}\left[F_t\left(C^{2\nu} - (X_{t\wedge T_0})^{2\nu}\right)^+\right] = C^{2\nu} E_1^{(4-\delta)}\left[F_t 1_{(\gamma_C \le t)}\right]$$

with $\gamma_C = \sup\{t < T_0 : X_t = C\}$.

(iii)$_\delta$ the time reversal result:
$(X_{(T_0-t)},\ t \le T_0)$ under $P_1^{(4-\delta)}$ is distributed as $(X_t, t \le g_1)$ under $P_0^{(\delta)}$.

3.3.2. The preceding results lead us to the following descriptions of $r_K^{(\delta)}(t)$ in terms of first and last passage times:

$$\begin{aligned} r_K^{(\delta)}(t) &= P_1^{(4-\delta)}(\gamma_k \le t \le T_0) && (3)_\delta \\ &= P_0^{(\delta)}(g_1 > t) - P_k^{(\delta)}(g_1 > t) && (4)_\delta \end{aligned}$$

where $k = 1/K^{1/(\delta-2)}$

3.3.3. Proof of Proposition 5:
It suffices to follow the arguments of the proof of Proposition 2, replacing everywhere dimensions 1 and 3 by dimensions $(4-\delta)$ and δ. □

3.3.4. Proof of Proposition 6:
Starting from $(3)_\delta$ we obtain:

$$\begin{aligned}
E\left[\exp\left(-\lambda\Lambda_K^{(\delta)}\right)\right] &= \left(\frac{\delta}{k^2}\right) E_1^{(4-\delta)}\left[\int_{g_k}^{T_0} dt\ e^{-\lambda t}\right] \\
&= \frac{\delta}{k^2\lambda}\left(E_1^{(4-\delta)}\left(e^{-\lambda g_k}\right) - E_1^{(4-\delta)}\left(e^{-\lambda T_0}\right)\right) \\
&= \frac{\delta}{k^2\lambda} E_1^{(4-\delta)}\left(e^{-\lambda T_0}\right)\left\{\frac{1}{E_0^{(\delta)}\left(e^{-\lambda T_k}\right)} - 1\right\} \\
&= \frac{\delta}{k^2\lambda} E_0^{(\delta)}\left(e^{-\lambda g_1}\right)\left\{\frac{1}{E_0^{(\delta)}\left(e^{-\lambda T_k}\right)} - 1\right\}
\end{aligned}$$

with the help of the time reversal result $(\mathrm{iii})_\delta$. Now, classical computations of Laplace transforms for first hitting times and last passage times of Bessel processes yield (see, e.g., Kent [4], Pitman-Yor [10]):

$$\begin{aligned}
E_0^{(\delta)}\left(e^{-\lambda g_1}\right) &= \frac{2}{\Gamma(\nu)}\left(\frac{\sqrt{2\lambda}}{2}\right)^{\nu} \mathcal{K}_\nu\left(\sqrt{2\lambda}\right) \\
E_0^{(\delta)}\left(e^{-\lambda T_k}\right) &= \frac{(\sqrt{2\lambda}k)^\nu}{2^\nu\Gamma(\nu+1)I_\nu(k\sqrt{2\lambda})}
\end{aligned}$$

where $\mathcal{K}_\nu$ and I_ν denote the usual modified Bessel functions with index ν. Finally, we have obtained the formula given in Proposition 6.

3.3.5. Proof of Proposition 4:
For $K = 0$, the result follows by scaling, as in dimension 3.
For $K > 0$,

(i) We shall use formula $(4)_\delta$ (to obtain the asymptotic result for $r_K^{(\delta)}(t)$, $t \to \infty$) together with the formula for the distribution of a last passage time of a transient diffusion (see, e.g., Pitman-Yor [10], as well as Salminen [12, 13]):

$$P_x\left(g_y \in dt\right) = -\frac{s'(y)a(y)}{2s(y)} p_t(x,y)dt.$$

In our present case, we have: $s(y) = -1/y^{(\delta-2)}$, so that: $-\frac{s'(y)}{s(y)} = \frac{(\delta-2)}{y}$; $a(y) = 1$. Thus we have:

$$P_x^{(\delta)}(g_y \in dt) = \left(\frac{\delta-2}{2y}\right) p_t^{(\delta)}(x,y)dt,$$

so that, going back to the expression $(4)_\delta$ for $r_K^{(\delta)}$, we get:

$$r_K^{(\delta)}(t) = \frac{\delta-2}{2}\int_t^\infty ds\ \left(p_s^{(\delta)}(0,1) - p_s^{(\delta)}(k,1)\right). \qquad (6)$$

Next, we shall use the following explicit formulae: (with $\nu = \frac{\delta-2}{2}$)

$$\begin{aligned}
p_t^{(\delta)}(x,y) &= \frac{1}{t}\left(\frac{y}{x}\right)^{\nu}\exp\left(-\frac{(x^2+y^2)}{2t}\right) I_\nu\left(\frac{xy}{t}\right) \\
p_t^{(\delta)}(0,y) &= \frac{1}{2^\nu t^{\nu+1}\Gamma(\nu+1)} y^{2\nu+1}\exp\left(-\frac{y^2}{2t}\right)
\end{aligned}$$

(see, e.g., Revuz-Yor [11], Chapter XI). Thus:

$$\begin{aligned}
& p_s^{(\delta)}(0,1) - p_s^{(\delta)}(k,1) \\
= \; & \frac{1}{2^\nu s^{\nu+1}\Gamma(\nu+1)} \exp\left(-\frac{1}{2s}\right) - \frac{1}{s}\frac{1}{k^\nu}\exp\left(-\frac{k^2+1}{2s}\right) I_\nu\left(\frac{k}{s}\right) \\
= \; & \exp\left(-\frac{1}{2s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{2^\nu s^\nu \Gamma(\nu+1)} - \frac{1}{k^\nu}\exp\left(-\frac{k^2}{2s}\right) I_\nu\left(\frac{k}{s}\right)\right) \\
= \; & \exp\left(-\frac{1}{2s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{k^\nu}\right)\left(\left(\frac{k}{s}\right)^\nu \frac{1}{2^\nu\Gamma(1+\nu)} - \exp\left(-\frac{k^2}{2s}\right) I_\nu\left(\frac{k}{s}\right)\right).
\end{aligned}$$

From (6), it now remains to study the asymptotic, as $t \to \infty$, of:

$$r_K^{(\delta)}(t) = \frac{\nu}{k^\nu}\int_t^\infty \frac{ds}{s}\exp\left(-\frac{1}{2s}\right)\left(\left(\frac{k}{s}\right)^\nu \frac{1}{2^\nu\Gamma(1+\nu)} - \exp\left(-\frac{k^2}{2s}\right) I_\nu\left(\frac{k}{s}\right)\right)$$

Making the change of variables: $s = \frac{1}{x}$, we get, from formula (6):

$$\begin{aligned}
r_K^{(\delta)}(t) \; = \; & \frac{\nu}{k^\nu}\int_0^{1/t}\frac{dx}{x}\exp\left(-\frac{x}{2}\right) \\
& \left((kx)^\nu \frac{1}{2^\nu\Gamma(1+\nu)} - \exp\left(-\frac{k^2 x}{2}\right) I_\nu(kx)\right).
\end{aligned}$$

Letting: $y = kx$, we get:

$$r_K^{(\delta)} = \frac{\nu}{k^\nu}\int_0^{k/t}\frac{dy}{y}e^{-\frac{y}{2k}}\left(\frac{y^\nu}{2^\nu\Gamma(1+\nu)} - e^{-\frac{ky}{2}} I_\nu(y)\right) \tag{7}$$

Now, we have (see, e.g., Lebedev [5]):

$$I_\nu(y) = \left(\frac{y}{2}\right)^\nu \left(\frac{1}{\Gamma(1+\nu)} + \sum_{j=1}^\infty \left(\frac{y}{2}\right)^{2j}\frac{1}{j!\Gamma(\nu+j+1)}\right).$$

Consequently:

$$\begin{aligned}
\frac{y^\nu}{2^\nu\Gamma(\nu+1)} - e^{-\frac{ky}{2}} I_\nu(y) \; = \; & \left(\frac{y}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}\left(1 - e^{-\frac{ky}{2}}\right) \\
& - e^{-\frac{ky}{2}}\left(\frac{y}{2}\right)^\nu\left(\sum_{j=1}^\infty\left(\frac{y}{2}\right)^{2j}\frac{1}{j!\Gamma(\nu+j+1)}\right) \\
\underset{y\to 0}{\sim} \; & \left(\frac{y}{2}\right)^\nu\left(\frac{1}{\Gamma(1+\nu)}\left(\frac{ky}{2}\right)\right).
\end{aligned}$$

Finally, going back to (7), we get:

$$\begin{aligned}
r_K^{(\delta)}(t) \; \sim \; & \left(\frac{\nu}{k^\nu}\right)\frac{1}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)\int_0^{k/t} dy\left(\frac{y}{2}\right)^\nu \\
\sim \; & \frac{1}{t^{\nu+1}}\left(\frac{k^2}{2^{\nu+1}(\nu+1)\Gamma(\nu)}\right).
\end{aligned}$$

Thus, we have obtained the asymptotic result in (i) of Proposition 4, with $C_K^{(\delta)} = \frac{k^2}{2^{\nu+1}(\nu+1)\Gamma(\nu)}$. Note that in the particular case $\delta = 3$, hence: $\nu = \frac{1}{2}$, we get: $C_K^{(3)} = \frac{1}{3}\frac{1}{\sqrt{2\pi}}\frac{1}{K^2}$ as claimed in Proposition 1.

(ii) Now, we study the asymptotic as $t \to 0$. The first result is easily obtained, using the convergence in L^1 of $\frac{1}{R_t^{\delta-2}}$ to 1, as $t \to 0$.

We now give the details in the case $K = 1$. As previously, we use:

$$r_1^{(\delta)}(t) = E_1^{(4-\delta)}\left(\left(1 - R_t^{2\nu}\right)_+ 1_{(t<T_0)}\right). \tag{8}$$

Now, $1 - R_{t\wedge T_0}^{2\nu},\ t \geq 0$, is a martingale under $P_1^{(4-\delta)}$, which may be written as:

$$N_t^{(\nu)} = (2\nu)\gamma_{\int_0^{t\wedge T_0} ds\left(R_s^{(2\nu-1)}\right)^2} \tag{9}$$

with $(\gamma_u, u \geq 0)$ a standard Brownian motion. We then write:

$$r_1^{(\delta)}(t) = E_1^{(4-\delta)}\left(\left(N_t^{(\nu)}\right)_+\right) - E_1^{(4-\delta)}\left(\left(N_t^{(\nu)}\right)_+ 1_{(T_0<t)}\right)$$

We shall show:

$$E_1^{(4-\delta)}\left(\left(N_t^{(\nu)}\right)_+\right) \sim \sqrt{t}(2\nu)\frac{1}{\sqrt{2\pi}} \tag{10}$$

as well as:

$$P_1^{(4-\delta)}(T_0 < t) = P_0^{(\delta)}(g_1 < t) = \mathrm{o}(\sqrt{t}). \tag{11}$$

Indeed, from (9), we deduce that the LHS of (10) is:

$$\begin{aligned}
&(2\nu)\sqrt{t}E\left[\left(\widetilde{\gamma}_{\frac{1}{t}\int_0^{t\wedge T_0} ds\left(R_s^{(2\nu-1)}\right)^2}\right)_+\right] \\
\sim\quad &(2\nu)\sqrt{t}E\left[(\widetilde{\gamma}_1)_+\right] \\
=\quad &(2\nu)\sqrt{t}\frac{1}{\sqrt{2\pi}}
\end{aligned}$$

where $(\widetilde{\gamma}_u,\ u \geq 0)$ denotes a standard Brownian motion.

Concerning (11), we use Getoor's result [3]

$$g_1 \overset{\text{(law)}}{=} 1/2\gamma_\nu,$$

where γ_ν denotes a gamma(ν) variable; thus:

$$\begin{aligned}
&P_0^{(\delta)}(g_1 < t) = P\left(\gamma_\nu > \frac{1}{2t}\right) \\
=\quad &\frac{1}{\Gamma(\nu)}\int_{1/2t}^{\infty} dx\ x^{\nu-1}e^{-x} \\
=\quad &\frac{1}{\Gamma(\nu)}e^{-(\frac{1}{2t})}\int_0^{\infty} dy\ e^{-y}\left(\frac{1}{2t} + y\right)^{\nu-1}.
\end{aligned}$$

For $\nu \geq 1$, this quantity is equivalent to: $\frac{1}{\Gamma(\nu)}e^{-(1/2t)}\left(\frac{1}{2t}\right)^{\nu-1}$ whereas, for $\nu < 1$, it is majorized by: $e^{-(1/2t)}$. In any case, we easily deduce (11).

Remark: Using (8) and (11), we see that:

$$r_1^{(\delta)}(t) \sim E_1^{(4-\delta)}\left(\left(1 - R_t^{2\nu}\right)_+\right).$$

It should be possible to use the explicit form of the semigroup density $\mathcal{P}_t^{\delta}(1, r)$, for $\delta' = 4 - \delta$, at least when $\delta < 4$, to derive (10) more directly.

3.3.6. A more direct proof of the estimate: $r_K^{(\delta)}(t) = \mathrm{O}\left(\frac{1}{t^{\nu+1}}\right)$, as $t \to \infty$. Similar to the case of dimension 3, we have, for every dimension $\delta > 2$:

$$r_K^{(\delta)}(t) \leq E_0^{(\delta)}\left(\left(\frac{1}{R_t} - K\right)^+\right) = E_0^{(\delta)}\left(\left(\frac{1}{\sqrt{t}R_1} - K\right)^+\right)$$

so that:

$$\begin{aligned} r_K^{(\delta)}(t) &\leq c\int_0^{1/\sqrt{t}K} dr\; r^{\delta-1} e^{-r^2/2}\left(\frac{1}{\sqrt{t}r}\right) \\ &= \frac{c}{\sqrt{t}}\int_0^{1/\sqrt{t}K} dr\; r^{2\nu} e^{-r^2/2} \\ &\leq \frac{c}{\sqrt{t}}\frac{1}{2\nu+1}\left(\frac{1}{\sqrt{t}K}\right)^{2\nu+1} \\ &= \frac{c}{(2\nu+1)K^{2\nu+1}}\left(\frac{1}{t^{\nu+1}}\right). \end{aligned}$$

4. Drawing the graphs of $(r_K^{(\delta)}(t), t \geq 0)$

4.1. In order to facilitate the drawing of these graphs, we need to use the simplest possible integral representations of these functions. We shall rely essentially upon formula (7) which was the key of our proof of Proposition 4:

$$r_K^{(\delta)}(t) = \frac{\nu}{k^\nu}\int_0^{k/t}\left(\frac{dy}{y}\right)e^{-y/2k}\left(\frac{y^\nu}{2^\nu\Gamma(1+\nu)} - e^{-\frac{ky}{2}}I_\nu(y)\right)$$

Using again the decomposition:

$$I_\nu(y) = \left(\frac{y}{2}\right)^\nu\frac{1}{\Gamma(1+\nu)} + \widetilde{I}_\nu(y)$$

$$\text{where:}\quad \widetilde{I}_\nu(y) = \left(\frac{y}{2}\right)^\nu\left(\sum_{j=1}^\infty\left(\frac{y}{2}\right)^{2j}\frac{1}{j!\Gamma(\nu+j+1)}\right),$$

we obtain:

$$r_K^{(\delta)}(t) = r_K^{(\delta)1}(t) - r_K^{(\delta)2}(t), \quad \text{with}$$

$$\begin{cases} r_K^{(\delta)1}(t) &= \frac{\nu}{k^\nu}\int_0^{k/t}\frac{dy}{y}\left(\frac{y}{2}\right)^\nu\frac{1}{\Gamma(\nu+1)}e^{-(y/2k)}\left(1 - e^{-\frac{ky}{2}}\right) \\ r_K^{(\delta)2}(t) &= \frac{\nu}{k^\nu}\int_0^{k/t}\frac{dy}{y}e^{-\frac{y}{2}\left(k+\frac{1}{k}\right)}\widetilde{I}_\nu(y). \end{cases}$$

4.2. In the case $\delta = 3$, a slightly different approach leads to the following:

$$r_K^{(3)}(t) = \frac{1}{k}\sqrt{\frac{2}{\pi}}\int_0^{1/\sqrt{t}} dy\left[e^{-y^2/2} - \frac{1}{2ky^2}\left\{e^{-\frac{(1-k)^2y^2}{2}} - e^{-\frac{(1+k)^2y^2}{2}}\right\}\right] \tag{12}$$

Thus, we see the importance of the function:

$$\varphi(A) \stackrel{\text{def}}{=} \sqrt{\frac{2}{\pi}}\int_0^A\frac{dy}{y^2}\left(1 - e^{-y^2/2}\right), \quad A \geq 0 \tag{13}$$

and we note that:

$$\varphi(A) = \widetilde{N}(A) - \frac{1}{A}\sqrt{\frac{2}{\pi}}\left(1 - e^{(-A^2/2)}\right) \tag{14}$$

where:

$$\widetilde{N}(A) = \sqrt{\frac{2}{\pi}}\int_0^A dy\; e^{-y^2/2}.$$

Indeed, from formula (12), we deduce:

$$r_K^{(3)}(t) = \frac{1}{k}\sqrt{\frac{2}{\pi}}\int_0^{1/\sqrt{t}} dy\; e^{-y^2/2} + \frac{1-k}{2k^2}\;\varphi\left(\frac{1-k}{\sqrt{t}}\right) - \frac{1+k}{2k^2}\;\varphi\left(\frac{1+k}{\sqrt{t}}\right) \tag{15}$$

From formula (15), we have:

$$\begin{aligned}
r_K^{(3)}(t) \quad &= \quad \frac{1}{k}\widetilde{N}\left(\frac{1}{\sqrt{t}}\right) + \frac{(1-k)}{2k^2}\varphi\left(\frac{1-k}{\sqrt{t}}\right) - \frac{(1+k)}{2k^2}\varphi\left(\frac{1+k}{\sqrt{t}}\right) && (16)\\
\text{(by (14))} \quad &= \quad \frac{1}{k}\widetilde{N}\left(\frac{1}{\sqrt{t}}\right) + \frac{1}{2k^2}\left(\varphi\left(\frac{1-k}{\sqrt{t}}\right) - \varphi\left(\frac{1+k}{\sqrt{t}}\right)\right)\\
&\qquad -\frac{1}{2k}\left(\varphi\left(\frac{1-k}{\sqrt{t}}\right) + \varphi\left(\frac{1+k}{\sqrt{t}}\right)\right)\\
&= \quad \frac{1}{k}\left\{\widetilde{N}\left(\frac{1}{\sqrt{t}}\right) - \frac{1}{2}\left[\widetilde{N}\left(\frac{1-k}{\sqrt{t}}\right) + \widetilde{N}\left(\frac{1+k}{\sqrt{t}}\right)\right.\right.\\
&\qquad \left.\left. -\left[\frac{\sqrt{t}}{(1-k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1-k)^2}{2t}}\right) + \frac{\sqrt{t}}{(1+k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1+k)^2}{2t}}\right)\right]\right]\right\}\\
&\qquad +\frac{1}{2k^2}\left\{\widetilde{N}\left(\frac{1-k}{\sqrt{t}}\right) - \widetilde{N}\left(\frac{1+k}{\sqrt{t}}\right)\right.\\
&\qquad \left. -\frac{\sqrt{t}}{(1-k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1-k)^2}{2t}}\right) + \frac{\sqrt{t}}{(1+k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1+k)^2}{2t}}\right)\right\}\\
&\equiv \quad \frac{1}{k}\left\{\widetilde{N}\left(\frac{1}{\sqrt{t}}\right) - \frac{1}{2}\left[\widetilde{N}\left(\frac{1-k}{\sqrt{t}}\right) + \widetilde{N}\left(\frac{1+k}{\sqrt{t}}\right)\right]\right\}\\
&\qquad +\frac{1}{2k^2}\left[\widetilde{N}\left(\frac{1-k}{\sqrt{t}}\right) - \widetilde{N}\left(\frac{1+k}{\sqrt{t}}\right)\right] + \frac{\sqrt{t}}{2k(1-k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1-k)^2}{2t}}\right)\\
&\qquad +\frac{\sqrt{t}}{2k(1+k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1+k)^2}{2t}}\right) - \frac{\sqrt{t}}{2k^2(1-k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1-k)^2}{2t}}\right)\\
&\qquad +\frac{\sqrt{t}}{2k^2(1+k)}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{(1+k)^2}{2t}}\right) && (17)
\end{aligned}$$

Formula (17) involves 5 terms with $\widetilde{N}$, and 4 terms with "exp". Thus, we write:

$$r_K^{(3)}(t) = \widetilde{r}_k(t) + r_k^{(\exp)}(t),$$

with

$$\begin{aligned}
\widetilde{r}_k(t) \quad &= \quad \frac{1}{k}\left\{\widetilde{N}\left(\frac{1}{\sqrt{t}}\right) - \frac{1}{2}\left[\widetilde{N}\left(\frac{1-k}{\sqrt{t}}\right) + \widetilde{N}\left(\frac{1+k}{\sqrt{t}}\right)\right]\right\}\\
&\qquad +\frac{1}{2k^2}\left[\widetilde{N}\left(\frac{1-k}{\sqrt{t}}\right) - \widetilde{N}\left(\frac{1+k}{\sqrt{t}}\right)\right] && (18)
\end{aligned}$$

$$
\begin{aligned}
\widetilde{r}_k^{(\exp)}(t) &= \sqrt{\frac{t}{2\pi}}\left\{\left(\frac{1}{k}-\frac{1}{k^2}\right)\frac{1}{(1-k)}\left(1-e^{-\frac{(1-k)^2}{2t}}\right)\right. \\
&\quad \left.+\left(\frac{1}{k}+\frac{1}{k^2}\right)\frac{1}{(1+k)}\left(1-e^{-\frac{(1+k)^2}{2t}}\right)\right\} \\
&= \sqrt{\frac{t}{2\pi}}\left\{-\frac{1}{k}\left(1-e^{-\frac{(1-k)^2}{2t}}\right)+\frac{1}{k}\left(1-e^{-\frac{(1+k)^2}{2t}}\right)\right\} \\
&= \sqrt{\frac{t}{2\pi}}\frac{1}{k}\left(\exp\left(-\frac{(1-k)^2}{2t}\right)+\exp\left(-\frac{(1+k)^2}{2t}\right)\right). \qquad (19)
\end{aligned}
$$

We note again the further simplification of (19):

$$
r_K^{(\exp)}(t) = \sqrt{\frac{2t}{\pi}}\left(\frac{1}{k}\right)\left(\exp\left(-\frac{1+k^2}{2t}\right)\right)\cosh\left(\frac{k}{t}\right). \qquad (19')
$$

4.3. We now present the graphs of $r_K^{\delta}(t)$ for various dimensions δ and strikes K. In fact, we have drawn two kinds of graphs:

(a) Figure 1: for each dimension δ, we consider $k = 1, \frac{1}{2}, \frac{1}{3}..., \frac{1}{10}$, and draw all the graphs together. We use formula (16) for $\delta = 3$, and formula (7) for $\delta = 5, 7, 9, 11, 13$ to draw the graphs.

(b) Figure 2: for each k, we draw the graphs for different dimensions. Here, we use formula (7) to draw graphs for all dimensions.

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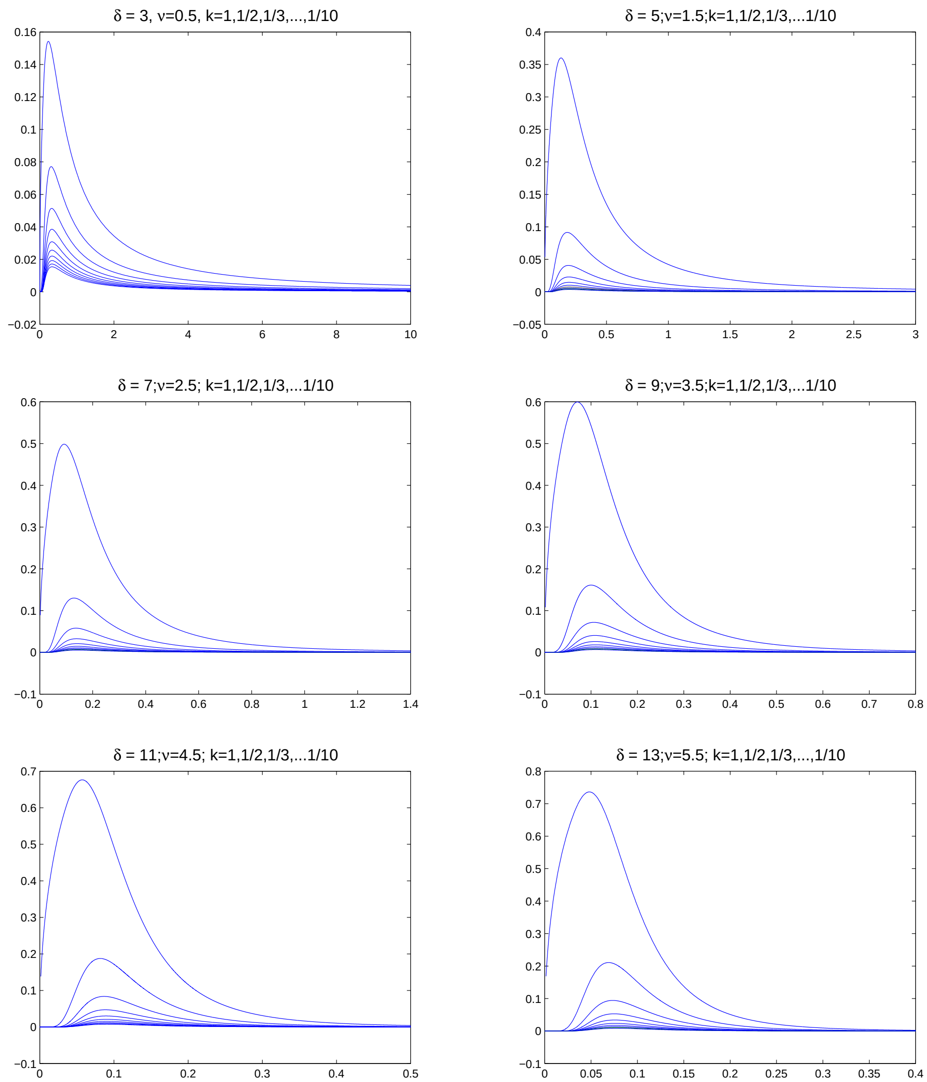

Figure 1. Varying k

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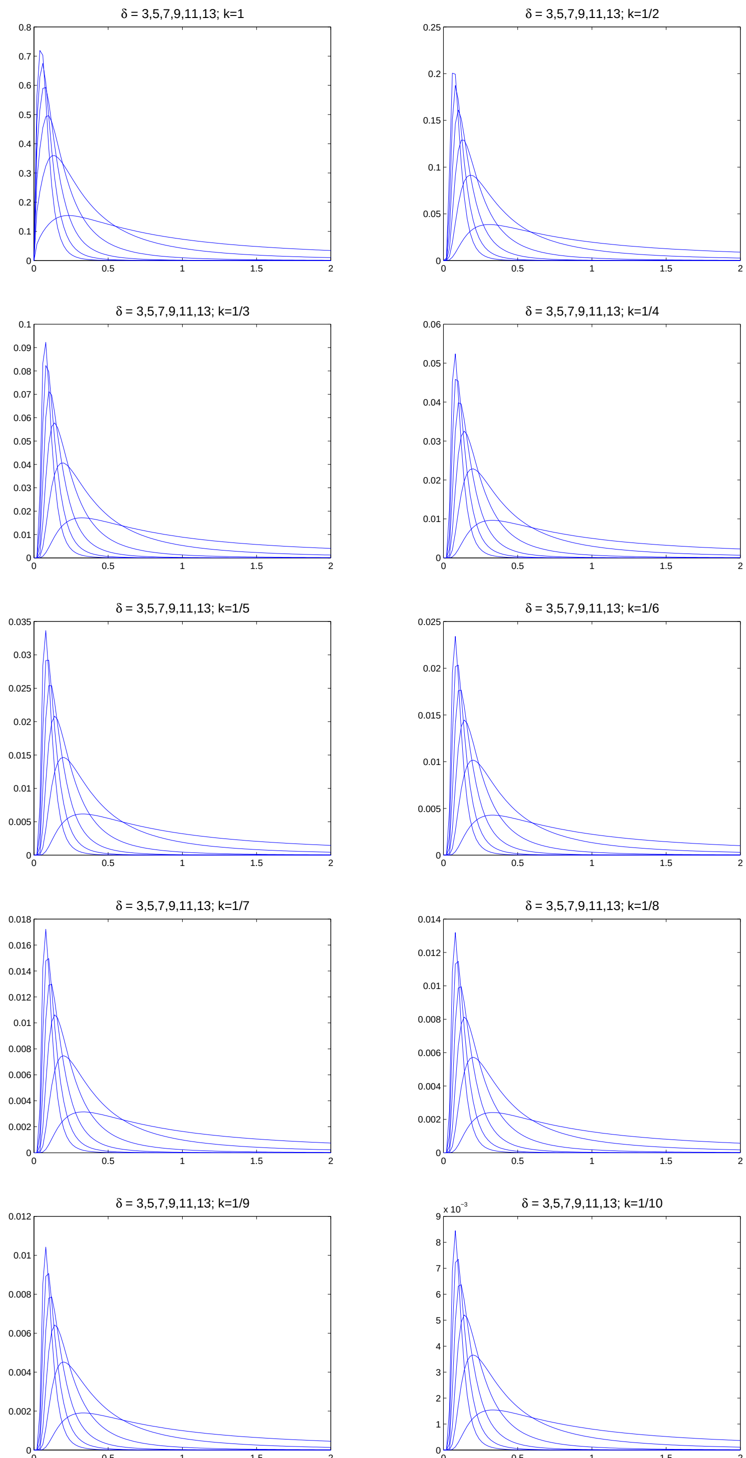

FIGURE 2. Varying δ